§1. Introduction

§2. $\Pi_n \rightsquigarrow n$

§3. $\Pi_n \rightsquigarrow (g, r)$

§4. $\mathrm{Out}(\Pi_n)$

$\boxed{\S 1}$

(g, r) : a pair of nonnegative integers s.t. $2g-2+r>0$

k : an alg closed field $ch(k) = p \geq 0$

X : a hyperbolic curve of type $(\underbrace{g}_{\text{genus}}, \underbrace{r}_{\text{the number of cusps}}) / k$

$\pi_1((-))$: th

Fact

of nonnegative
s.t. $2g-2+r>0$

closed field

$ch(k) = p \geq 0$

bolic curve of type (g,r) / k

genus

the number of cusps

$\pi_1((-))$: the étale fund gp of $(-)$

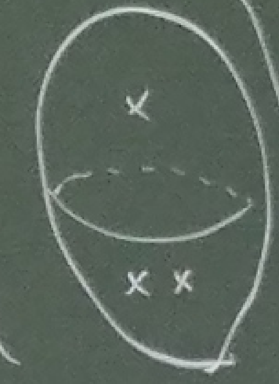
<u>Fact</u> If $p=0$, then

$$\pi_1(X) \cong \langle \alpha_1 \cdots \alpha_g, \beta_1 \cdots \beta_g, \gamma_1 \cdots \gamma_r \mid [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \gamma_1 \cdots \gamma_r = 1 \rangle^{\wedge}$$

prof comp

<u>Fact</u> implies that one cannot reconstruct (g,r) from $\pi_1(X)$

<u>e.g.</u>

$\pi_1($  $) \cong \widehat{\mathbb{Z} * \mathbb{Z}} \cong \pi_1($  $)$

(0,3) (1,1)

Rem Suppose that $p>0$.

Then one can reconstruct

(g, r) from $\pi_1(X)$

i.e., "$\pi_1(X^\circ) \cong \pi_1(X^\bullet)$

$\Rightarrow (g^\circ, r^\circ) = (g^\bullet, r^\bullet)$"

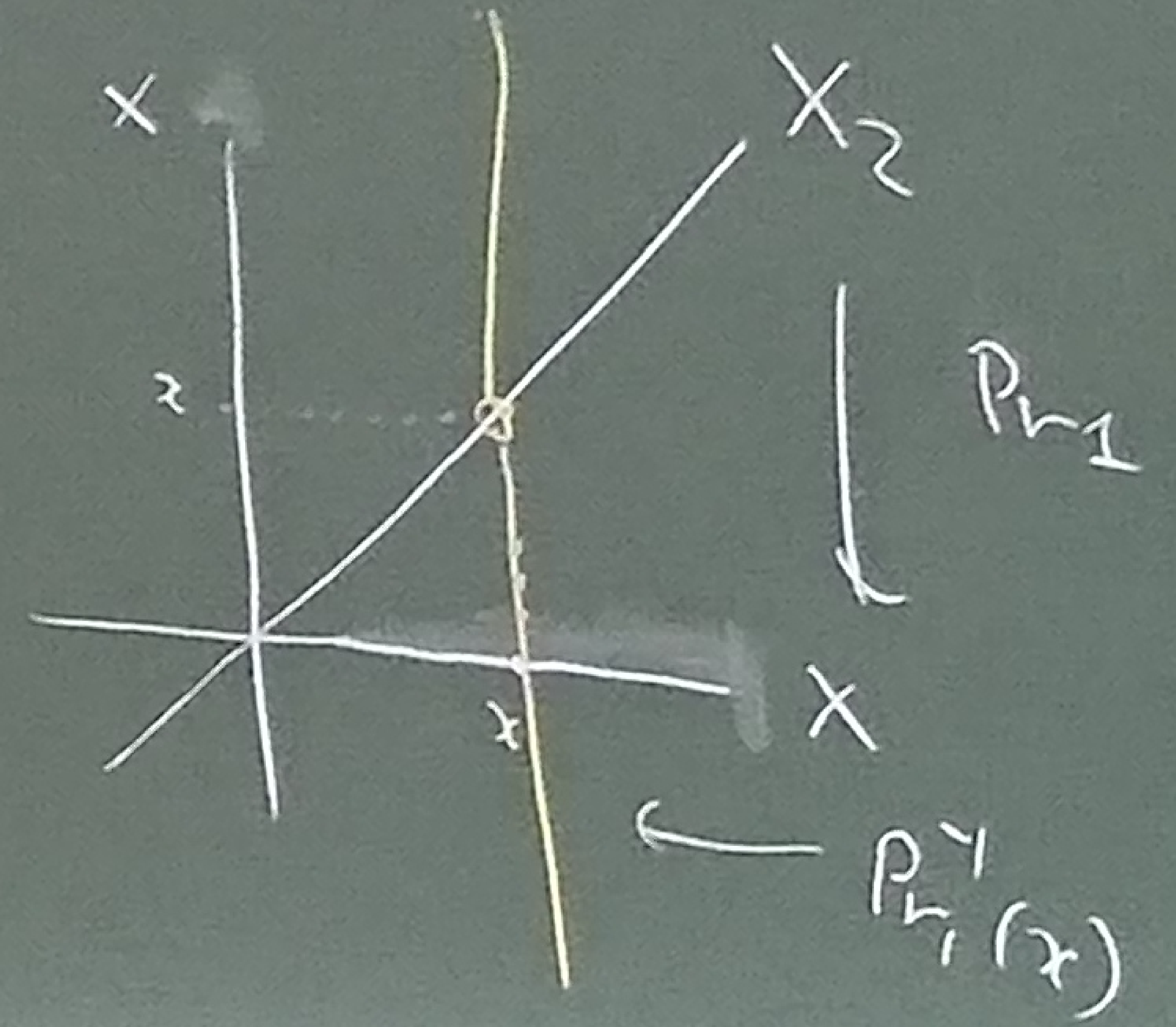
[Tamagawa] (1999) On the fundamental groups of curves over algebraically closed fields of ch >0

$n \in \mathbb{Z}_{>0}$

$X_n := \{ (x_1 \cdots x_n) \in \overbrace{X \times_k \cdots \times_k X}^{n} \mid x_i \neq x_j \ (\forall i \neq j) \}$

the n-th configuration space of X

e.g. $(n=2)$



$\overbrace{\times_k \cdots \times_k}^{n} X$

$x_j \ (i \neq j) \}$

ion space of X

X_2

$\downarrow pr_1$

X

$\leftarrow pr_1^{-1}(x)$

$\boxed{\text{Thm A}}$ $\square \in \{\circ, \bullet\}$

$X^\square$: a hyperbolic curve of type $(g^\square, r^\square)$ / an alg closed field of ch $= 0$

$X^\square_{n^\square}$: the $n^\square$-th config sp of $X^\square$

Suppose that $\exists$ isom $\alpha : \pi_1(X^\circ_{n^\circ}) \xrightarrow{\sim} \pi_1(X^\bullet_{n^\bullet})$ of prof gps.

Then $n^\circ = n^\bullet =: n$

Moreover, if $n \geq 2$, then

$(g^\circ, r^\circ) = (g^\bullet, r^\bullet)$

Rem (i) The discrete, pro-ℓ ← prime cases also hold

(ii) "$n^\circ = n^\bullet$" also follows from a result of Sawada.

$\boxed{\S 2}$

Today, we consider the pro-ℓ case of $\boxed{\text{Thm A}}$

In the following §2, §3,

$\Pi_{(-)} := \pi_1((-))^{(\ell)}$

maximal pro-ℓ quot

<u>Lem 1</u>

$G \cong \pi_1({}^{\exists}\text{hyperbolic curve}/\mathbb{C})^{(\ell)}$

$H \subseteq G$: an abelian closed subgp

Then $H \cong \{1\}$ or $\cong \mathbb{Z}_\ell$

<u>affine case</u> $\rightsquigarrow$ G : a free pro-ℓ

Thus, H is also free pro-ℓ

Since H is abelian, $H \cong \{1\}$ or $\mathbb{Z}_\ell$.

<u>proper case</u> : similar

<u>Lem 2</u>

X : a

$)^{(\ell)}$

ed subgp

$\cong \mathbb{Z}_\ell$

free pro-ℓ

ro-ℓ

$1\}$ or $\mathbb{Z}_\ell$.

---

lem 2

X : a hyperbolic curve / an alg closed field of ch $=0$

X_n : the n-th config sp of X

$\Pi_n := \Pi_{X_n}$

If there exists a closed subgp

$H \subseteq \Pi_n$ s.t. $H \simeq \mathbb{Z}_\ell^{\oplus s}$,

then $s \leq n$

In the case where $n=1$

lem 2 follows from lem 1

Suppose that $n \geq 2$ and the induction hypothesis is in force.

$X_n \longrightarrow X$: forgetting

the factors

labels $\neq n$

$p : \Pi_n \twoheadrightarrow \Pi_1$

By lem 1, $p(H) \cong \{1\}$ or $\cong \mathbb{Z}_\ell$

Thus, $H' := H \cap \ker p$

$(\subseteq \ker p)$

satisfies

$H' \cong \mathbb{Z}_\ell^{\oplus s}$ or $\cong \mathbb{Z}_\ell^{\oplus s-1}$

But since $\ker p$ may be

identified with

Π_1 ($(n-1)$-st config sp of $^{\exists}$ hyp curve $^{(\ell)}$ / $^{\exists}$ an alg closed field of ch $= 0$)

$s-1$

(ℓ)

gp curve

am alg closed

field of ch=0

we have

$s \le n-1$ or $s-1 \le n-1$

$\therefore s \le n$ //

Prop 3

$$n = \max \left\{ m \in \mathbb{N} \;\middle|\; \exists \text{ closed subgp } \mathbb{Z}_\ell^{\oplus m} \cong \ \le \Pi_n \right\}$$

$\because$ By $\boxed{\text{lem 2}}$, To verify prop 3, it suffices to show that

$\boxed{\exists \text{ closed subgp } H \le \Pi_n \text{ s.t. } H \cong \mathbb{Z}_\ell^{\oplus n}}$

But, by using log geometry ← i.e. cusps and nodes of successive families of log stable curves

we can find such a subgp //